(1) Let

$$A = \{x | x \text{ is a whole number between 2 and 10}\}$$

and

$$B = \{x | x \text{ is an even whole number bigger than 2 and less than 10}\}.$$

Which statement is **false**?

Solutions: We can write A as $\{2, 3, 4, 5, 6, 7, 8, 9, 10\}$ and B as $\{4, 6, 8\}$. Then

$$A \cap B = \{4, 6, 8\}$$

Hence, A and B are not disjoint (recall, two sets are disjoint if their intersection is empty). The false answer is: A and B are disjoint.

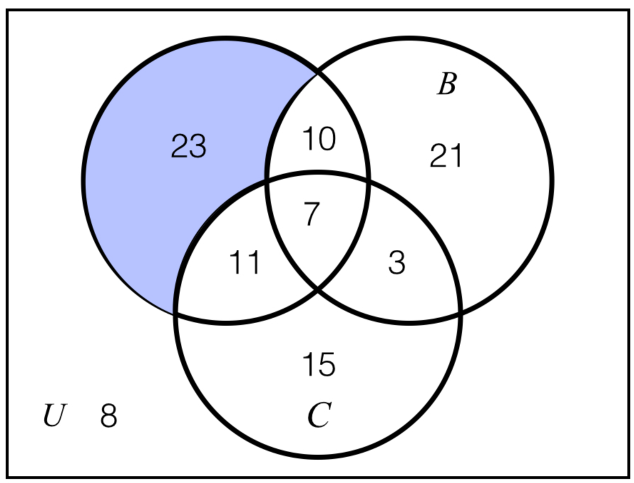
(2) Let A and B be sets such that $n(A) = 16$, $n(B) = 23$ and $n(A \cup B) = 32$. What is $n(A \cap B)$?

Solutions: Using the formula $n(A \cup B) = n(A) + n(B) - n(A \cap B)$, we get

$$32 = 16 + 23 - n(A \cap B)$$

Solving for $n(A \cap B)$, we get $n(A \cap B) = 7$.

(3) Let A, B, C be sets in some universe set U. The Venn diagram below shows the number of elements in each region of the diagram. What is $n(A \cap (B \cup C)')$?



Solutions: The answer is 23. Area shaded above.

(4) The FCC (Federal Communications Commission) wants to issue a unique code, consisting of a string of k letters (a through z) and numbers (0 through 9), to each of 65, 000, 000 devices. What's the smallest choice of k that makes this possible?

Solutions: There are 36 total digits we can use (26 letters and 10 numbers). Lets compute how many different codes can we get if $k = 5$. For each of the 5 digits of the code, we have 36 options, hence we have 36^5 codes, which is $60,466,176$ codes. Since we need $65,000,000$, this is not enough.

If $k = 6$, we would then have 36^6 codes (same analysis as above), which is $2,176,782,336$. Since we only need $65,000,000$, then the answer is $k = 6$.

(5) Compute $P(21, 5) \cdot 3!$.

Solutions: $P(21, 5) = 21 \cdot 20 \cdot 19 \cdot 18 \cdot 17 = 2,441,880$ and $3! = 3 \cdot 2 \cdot 1 = 6$. Hence,

$$P(21, 5) \cdot 3! = 14,651,280.$$

(6) A deli offers 4 different types of bread, 5 types of meat and 8 types of vegetables. I want a sandwich that has bread, one meat and two vegetables. How many options do I have?

Solutions: We are doing 3 activities. First, selecting a bread, **then** selecting a meat and **then** selecting 2 vegetables. Since the order in which we choose the vegetables does not matter (whether we choose tomato and lettuce or lettuce and tomato, we will get the same sandwich at the end), then there are $C(4, 1) \cdot C(5, 1) \cdot C(8, 2) = 4 \cdot 5 \cdot 28 = 560$ ways of selecting a sandwich.

(7) Mario the plumber receives 10 calls from residents of South Bend, concerning burst pipes. He has resources to deal with at most three of the calls. In how many ways can he choose at most three calls to deal with, if the order in which he responds to the calls matters? [Note: he might choose to respond to 0 of the calls].

Solutions: Notice that since Mario can choose at most 3 of the calls, he can choose 0 calls **OR** 1 call **OR** 2 calls **OR** 3 calls. Since the order in which he chooses the calls matters, then we have

$$P(10, 0) + P(10, 1) + P(10, 2) + P(10, 3)$$

ways of choosing the calls. This is, 821 ways.

(8) A student council committee has 7 reps from Carroll Hall, 6 reps Badin Hall and 10 from Pasquerilla East Hall. In how many ways can a sub-committee of three reps be formed, if all three must be from the same hall?

Solutions: Since all 3 students have to be from the same hall, we can choose 3 students from Carroll Hall **OR** Badin Hall **OR** Pasquerilla East Hall. If we choose them from Carroll, we have $C(7, 3) = 35$ ways of selecting them (remember the order does not matter when choosing a committee since there is no distinction between members). If we choose them from Badin, we have $C(6, 3) = 20$ ways of selecting them and if we choose them from Pasquerilla East, we have $C(10, 3) = 120$ ways of selecting them. In total, we have

$$C(7, 3) + C(6, 3) + C(10, 3) = 35 + 20 + 120 = 175$$

ways of selecting the committee.

(9) My Combinatorics class has 9 students. I have three different final projects in mind for the class. In how many ways can I split the class into 3 groups of three people each, assigning one group to do the first project, one to do the second, and one to do the third, and choose a group

leader for each group?

Solutions: This is a partition problem. Since we are assigning different projects to different groups, the order of the groups matters and hence this is an ordered partition. The number of ways in which we can partition a 9 element set into 3 ordered subsets of 3 elements is $\binom{9}{3,3,3}$. However, we must also choose a leader for each of the 3 groups. We have 3 options **for each group**, hence we multiply by $3 \cdot 3 \cdot 3$. The final answer is

$$\binom{9}{3,3,3} \cdot 3^3.$$

(10) There are 14 men and 9 women in the Jackson family and 12 men and 18 women in the Jones family. If during a charity event, every woman shakes hands with all other women and every man shakes hands with all other men, how many handshakes take place?

Solutions: Lets first count the number of handshakes between women. There are 27 total women and they all shake hands with each other. Notice that counting the number of hand shakes is equivalent to counting the number of ways in which we can choose 2 women (think that when you choose 2 women, you are counting the hand shake between them). Hence, the order in which we choose the women does not matter and we have $C(27,2)$ handshakes between women. Similarly, we do the same for men and we get $C(26,2)$ handshakes between men. Therefore, there are $C(27,2) + C(26,2)$ total handshakes.

(11) A Notre Dame club has 53 students, of which 18 are juniors, 14 sophomores and 21 freshmen. For part (c) of this problem, you may leave your answer in terms of mixtures of combinations and permutations (i.e. $C(n,r)$ and $P(n,r)$ for appropriate n and r) if you choose.

(a) In how many ways can you choose a committee of 4 people?

There are 53 total students and we will choose 4 of them, where the order between them does not matter (in a committee order does not matter since there is no distinction between members). Hence, there are $C(53,4) = 292,825$ ways of choosing the committee.

(b) If there has to be exactly 2 juniors, in how many ways can you choose the committee?

Since there has to be **exactly** 2 juniors, then we are doing 2 activities. First, choosing the 2 juniors and **then** choosing the other 2 members (which cannot be juniors). There are $C(18,2)$ ways of selecting 2 juniors and $C(35,2)$ ways of selecting the other two members. Therefore, there are

$$C(18,2) \cdot C(35,2) = 91,035$$

ways of selecting the committee.

Note: Some people wrote $C(51,2)$ ways of selecting the other two members, but if you choose from the remaining 51 students (after you select the 2 juniors), you may select more juniors and we are required to have exactly 2 juniors.

(c) If I need to have at least one representative from each class, this is, at least one junior, sophomore and freshman, in how many ways can you choose the committee?

Since we need to have one representative from each class, we can choose either 2 juniors, 1 sophomore, 1 freshman (call this case 1) **OR** 1 juniors, 2 sophomore, 1 freshman (call this case 2) **OR** 1 juniors, 1 sophomore, 2 freshmen (call this case 3). There are $C(18,2)\cdot C(14,1)\cdot C(21,1)$ ways of selecting the committee in case 1. There are $C(18,1)\cdot C(14,2)\cdot C(21,1)$ ways of selecting the committee in case 2. There are $C(18,1)\cdot C(14,1)\cdot C(21,2)$ ways of selecting the committee in case 3. Hence, there are

$$(18,2)\cdot C(14,1)\cdot C(21,1) + (18,1)\cdot C(14,2)\cdot C(21,1) + (18,1)\cdot C(14,1)\cdot C(21,2)$$

ways of selecting the committee satisfying the above conditions.

Note: Some people wrote $C(18,1)\cdot C(14,1)\cdot C(21,1)\cdot C(50,1)$ as the solution. However, you are over counting. For example, assume Peter and John are juniors, Maria is a sophomore and Daniel is a freshman. Then the selections:

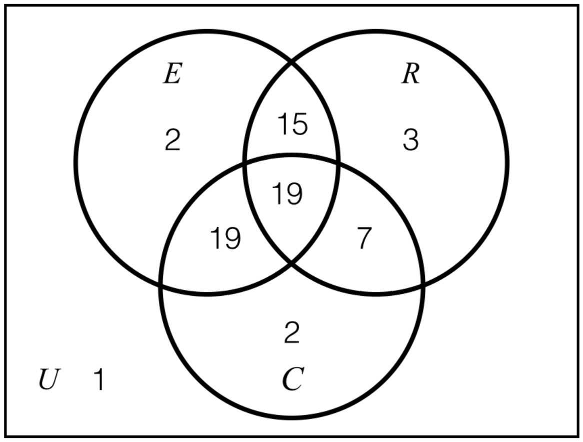
Peter (selected as the junior)	**John (selected as the junior)**
Maria (selected as the sophomore)	**Maria (selected as the sophomore)**
Daniel (selected as the freshman)	**Daniel (selected as the freshman)**
John (selected from the 50 remaining students as the last member)	**Peter (selected from the 50 remaining students as the last member)**

are being counted differently. The problem arises since you are selecting elements from the same set (say the set of junior students) but selecting them as if their order matters. For example, if we were to select two juniors (from the set of 18 juniors) and the order of the selection does not matter, it is incorrect to say that we can do this in $C(18,1)\cdot C(17,1) = 18\cdot 17$ ways. The correct result is $C(18,2) = 18\cdot 17/2$.

(12) The following three yes/no questions were posed to a class of 68 students in a survey:
- (i) Do You like rap music?
- (ii) Do You like classical music?
- (iii) Do You like 80's music?

The results showed that 44 liked rap music, 47 liked classical music and 55 liked 80's music. Nineteen students liked all three types of music, 7 liked rap and classical but not 80's, 19 liked classical and 80's but not rap, and 15 liked rap and 80's but not classical.

(a) Present the data given above on a Venn diagram, where R denotes the set of students who like rap, C denotes the set of students who like classical and E denotes the set of students who like 80's music.



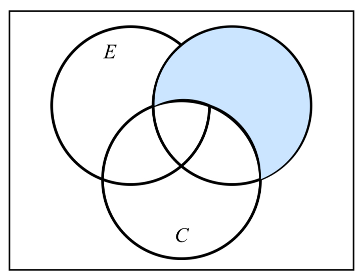
(b) How many students don't like any of the above music types?

1 student (the one outside the three circles).

(c) If a student is in the set $E \cap (R \cup C)'$, what answers did they give to questions (i), (ii) and (iii)?

Since they are in the complement of both R and C, they must have answered NO to both (i) and (ii). Since they are in E, they must have answered YES to (iii).

(d) Shade in the part of the Venn diagram on the right that corresponds to those students who answered "yes" to question (i) and "no" to question (ii).



(13) My after-hours access code for the Hayes-Healy building consists of 5 letters, to be entered in a particular order. I've forgotten the code, but I do remember that it only uses letters from the phrase "ACCESS CODE" (so it might have two S's, but no more, and it can have at most one A, etc.).

(a) If I also remember that the code has no repeated letters, how many possible codes are there?

There are 6 different letters which are: A, C, E, S, O, D. Since we cannot repeat letters and the order in which we select the letters matters, there are $P(6, 5) = 720$ different codes.

(b) If instead I remember that the code has two C's, and no other repeated letters, how many possible codes are there?

Notice that if the two C's go into the first two digits (or positions), then there are $P(5, 3) = 60$ ways of filling the other 3 digits (or positions). However, the 2 C's can go in any two positions.

Since there are $C(5,2) = 10$ different ways of arranging the 2 C's, then there are $10 \cdot P(5,3) = 600$ ways of selecting a code with exactly 2C's.

Another way of interpreting the problem is first selecting which 5 letters you will use, and then ordering them. In this analysis, since you know that you will use 2 C's, there are $C(5,3)$ ways of selecting the other 3 letters (not taking into consideration the order in which we choose them). Then, we have to order them, so there are 5! ways of ordering them. However, interchanging the 2 C's will not make a new code, so then we would have to divide by 2!. The answer is then $C(5,3) \cdot 5!/(2!) = 10 \cdot 120/2 = 600$.

Note: A lot of people wrote $P(5,3)$ as the answer, assuming that the 2 C's will go in the first two positions (which is not necessarily the case).

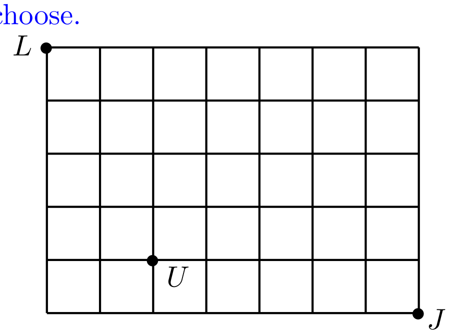
(c) If I remember that the code has no repeated letters, and that it begins and ends with a vowel, how many possible codes are there?

Since the code begins and ends with vowels (but we cannot use the same vowel twice), there are $P(3,2) = 6$ ways of selecting the vowels that go in the beginning and end of the code (taking the order into consideration). Then, we have 4 letters available to use in the middle three digits of the code, so there are $P(4,3) = 24$ ways of selecting those 3 digits (taking order into consideration). Hence, there are $P(3,2) \cdot P(4,3) = 144$ different ways of selecting the code.

(d) If I remember that the code has no repeated letters, that it begins and ends with a vowel, and that all the remaining letters are consonants, how many possible codes are there?

Since the code begins and ends with vowels (but we cannot use the same vowel twice), there are $P(3,2) = 6$ ways of selecting the vowels that go in the beginning and end of the code (taking the order into consideration). Now, the middle 3 digits of the code are consonants and there are 3 possible consonants we can use (C, S,D), hence there are $P(3,3) = 6$ ways of selecting those 3 digits (taking order into consideration). Hence, there are $P(3,2) \cdot P(3,3) = 36$ different ways of selecting the code.

(14) All parts of this problem refer to the following city map, where we have marked the local university (U) and the houses of Luis (L) and John (J). For this problem, you may leave your answer in terms of mixtures of combinations and permutations (i.e. $C(n,r)$ and $P(n,r)$ for appropriate n and r) if you choose.



(a) In how many ways can Luis walk from his house (L) to the university (U), in as few blocks as possible (6)? (Ignore "J" at this point.)

Selecting a path from L to U is equivalent to writing an ordered sequence of 6 letters made of 4 S's and 2 E's, where an S means you are walking one block south and an E means you are walking one block east. For example $SESSSE$ means you will walk one block south, then one east, then 3 blocks south and then one block east.

Notice that by knowing which two blocks you will walk in the east direction, you will know when to walk south (if you are not walking east, then you must walk south). Hence, counting the number of paths from L to U is equivalent of counting in how many ways can you put two E's in a six letters sequence. There are then $C(6, 2) = 15$ ways in which one can walk from L to U. Equivalently, you could have counted the ways of putting 4 S's in a six letter sequence, which is $C(6, 4) = 15$.

(b) In how many ways can Luis walk from his house (L) to John's house (J) stopping first at the university (U), in as few blocks as possible (12)?

We are doing two activities, first, walking from L to U, and then from U to J. There are $C(6, 2) = 15$ ways of walking from L to U (from part (a)) and $C(6, 1) = 6$ ways of walking from U to J. Hence, there are $15 \cdot 6 = 90$ ways of walking from L to J stopping first at U.

(c) In how many ways can Luis walk from his house (L) to John's house (J) **without** passing by the university (U), in as few blocks as possible (12)?

Notice that the number of ways of going from L to J without stopping at U is the number of ways of going from L to J minus the number of ways of going from L to J stopping at U. Since there are $C(12, 5) = 792$ ways of going from L to J (same analysis as in part (a)) and 90 ways of going from L to J stopping at U (answer to part (b)), then there are $792 - 90 = 702$ ways of going from L to J without stopping at U.